Consider a cylindrical jet as sketched in Fig. 1. For purposes of analysis, as both Rayleigh and Weber did, we consider a particular form of perturbation from the equilibrium cylindrical jet given by

$$r_s = a + \alpha(t)\cos kz \tag{3}$$

where a is the radius of the unperturbed cylinder, $\alpha(t)$ is a small time-dependent disturbance amplitude, and $k = 2\pi/\lambda$ is the wavenumber, λ being the wavelength of the perturbation. Cylindrical polar coordinates r, θ, z are being used. For simplicity, we assume no angular disturbance (θ-dependency).

For the case $R = 0$, the flow is irrotational and, as derived in the previous note,[1] the velocity components are

$$\begin{aligned} u_r &= \dot{\alpha}[I_1(kr)/I_1(ka)]\cos kz \\ u_z &= w_0 - \dot{\alpha}[I_0(kr)/I_1(ka)]\sin kz \end{aligned} \tag{4}$$

where w_0 is the velocity of the undisturbed jet and the dot denotes a time derivative. For this case, evaluation of Eq. (1) led to the result

$$\ddot{\alpha} = \alpha\left\{\frac{\gamma}{\rho a^3}\eta(1-\eta^2)\frac{I_1(\eta)}{I_0(\eta)}\right\} \tag{5}$$

where $\eta = ka$ and γ is the surface tension of the fluid. For $\eta < 1$, we have exponential solutions of the form $\alpha = \alpha_0 e^{\omega t}$, where

$$\omega^2 = (\gamma/2\rho a^3)(1-\eta^2)\eta^2 \tag{6}$$

and the approximation $I_1(\eta)/I_0(\eta) = \eta/2$ has been used.

By inspection of the $\mu = 0$ case [Eq. (4)], it is seen that u_z is only weakly dependent on r. Therefore, as an approximation, assume u_z is a function of z only. Then from the continuity equation

$$\partial u_z/\partial z + (1/r)(\partial/\partial r)(r u_r) = 0 \tag{7}$$

with the conditions that u_r be finite along the axis and that

$$(u_r)_{r=a} = \dot{\alpha}\cos kz \tag{8}$$

we have

$$\begin{aligned} u_r &= (r/a)\dot{\alpha}\cos kz \\ u_z &= w_0 - (2\dot{\alpha}/ka)\sin kz \end{aligned} \tag{9}$$

Now for both the inviscid and viscous free cylindrical jet

$$\tau_{rr} = -\left(\frac{\gamma}{r_s} + \frac{\gamma}{R}\right) = -\frac{\gamma}{a} + \frac{\alpha\gamma}{a^2}(1-k^2a^2)\cos kz \tag{10}$$

where $1/R$ is the curvature in the osculating plane. Further, for the freejet for $i \neq j$, $\tau_{ij} = 0$ at $r = a$, i.e. there are no surface shears. Selecting the stationary control volume over which to perform the integrations as the unperturbed jet; i.e., a cylinder with radius a, we can then evaluate the terms in the integral energy equation to be

$$\int_S \tau_{ij} n_j u_i \, dS = \frac{2n\pi^2\gamma\alpha\dot{\alpha}}{ka}(1-k^2a^2) \tag{11}$$

$$\int_S \frac{1}{2}\rho u_j n_j u_i u_i \, dS = 0 \tag{12}$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right) dV = \frac{n\pi^2\rho\dot{\alpha}\ddot{\alpha}}{2k^3}(8+k^2a^2) \tag{13}$$

$$\int_V R\,dV = \int_V 2\mu\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)^2\right] dV = \frac{n\pi^2\mu\dot{\alpha}^2}{2k}(24+k^2a^2) \tag{14}$$

The z integration has been taken over an even number of wavelengths, $z = 0$ to $2\pi n/k$, where n is an integer. Thus from Eqs. (11–14), the integral energy equation leads, after cancellation, to the differential equation

$$\ddot{\alpha} + \frac{\mu k^2}{\rho}\left(\frac{24+\eta^2}{8+\eta^2}\right)\dot{\alpha} - \frac{4\gamma\eta^2}{\rho a^3}\frac{(1-\eta^2)}{(8+\eta^2)}\alpha = 0 \tag{15}$$

For $\eta < 1$, the jet is unstable and for exponentially growing solutions $e^{\omega t}$, we have from Eq. (15)

$$\omega^2 + \frac{3\mu k^2\omega}{\rho} - \frac{\gamma}{2\rho a^3}(1-\eta^2)\eta^2 = 0 \tag{16}$$

Thus the effect of viscosity is to dampen the instability. However, only in the limit of infinite viscosity is the instability removed. It is further to be noted that since R is positive definite, a damping will always result, regardless of the exact choice of the velocity components. The damping coefficient $3\mu k^2/\rho$ is exactly the result obtained by Weber.[3] Note that for $\mu = 0$, Eq. (16) reduces to the Rayleigh result, Eq. (6), in the approximation $I_1(\eta)/I_0(\eta) = \eta/2$. For a very viscous jet, such that $(3\mu k^2/2\rho)^2 \gg (\gamma/2\rho a^3)$, the solution is

$$\omega = (\gamma/6\mu a)(1-\eta^2) \tag{17}$$

Longer wavelengths are thus more unstable for the viscous jet.

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Penetration of Particles Injected into a Constant Cross Flow

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Nomenclature

C_D = drag coefficient
D = particle diameter
$F(\alpha)$ = function given by Eq. (11)
Re = Reynolds number
t = time
u = velocity in the direction of the gas flow
v = velocity across the gas flow
y = lateral distance from injection point
Y = y_{max} = particle penetration
α = parameter defined by Eq. (8)
μ = dynamic viscosity
ρ = density

Subscripts

o = at injection point
g = gas
p = particle
st = based on Stokes drag

Introduction

In the analysis of gas-particle flow for combustion chambers and other applications, it often becomes necessary to compute the penetration of particles injected into a cross flow. For low Reynolds numbers (less than about one), Stokes drag ($C_D = 24/Re$) can be applied. The relationship between the drag force

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on a particle and its slip velocity, then, is linear, and the penetration of the particle does not depend on the velocity of the cross flow. At higher Reynolds numbers, the drag coefficient becomes a more complicated function of the Reynolds number, and particle penetration is affected by the cross flow because of the nonlinear relationship between drag and slip velocity. Various empirical correlations between the "standard" drag coefficient and the Reynolds number can be found in the literature, and the equation of motion, in general, must be solved numerically. Particle trajectories were computed in this manner by Brandt and Perini.[1] Some years ago, Putnam[2] pointed out that the relationship

$$C_D = (24/Re)(1+\tfrac{1}{6}Re^{2/3}) \tag{1}$$

is a good approximation for Reynolds numbers up to about 1000 and may lead to equations that can be solved in closed form. This expression for C_D apparently was first suggested in 1934 by Klyachko.[3] Putnam analyzed the penetration of particles into a stagnant gas. This approach is extended here to derive a convenient closed-form relationship for particle penetration into a constant cross flow.

Analysis

The motion of a particle in the direction of the flow, and at right angles to it, is given by

$$du_p/dt = C_D Re(3\mu/4\rho_p D^2)(u_g-u_p) \tag{2}$$

and

$$dv_p/dt = -C_D Re(3\mu/4\rho_p D^2)v_p \tag{3}$$

where the drag coefficient is given by Eq. (1) and the Reynolds number by

$$Re = \rho D[(u_g-u_p)^2+v_p{}^2]^{1/2}/\mu \tag{4}$$

Division of Eq. (2) by Eq. (3) yields

$$du_p/dv_p = -(u_g-u_p)/v_p \tag{5}$$

Since $u_g = \text{const}$, this equation can be integrated and, for the initial conditions $u_p = u_{p_o}$ and $v_p = v_{p_o}$, leads to the relationship

$$u_g-u_p = (u_g-u_{p_o})v_p/v_{p_o} \tag{6}$$

which does not depend on the selected drag coefficient.

Since $dv_p/dt = v_p dv_p/dy$, combination of Eqs. (1, 3, 4, and 6) yields

$$dv_p/dy = -(18\mu/\rho_p D^2)[1+\alpha(v_p/v_{p_o})^{2/3}] \tag{7}$$

where α is a constant given by

$$\alpha = \frac{1}{6}\left(\frac{\rho_g D v_{p_o}}{\mu}\right)^{2/3}\left[1+\left(\frac{u_g-u_{p_o}}{v_{p_o}}\right)^2\right]^{1/3} \tag{8}$$

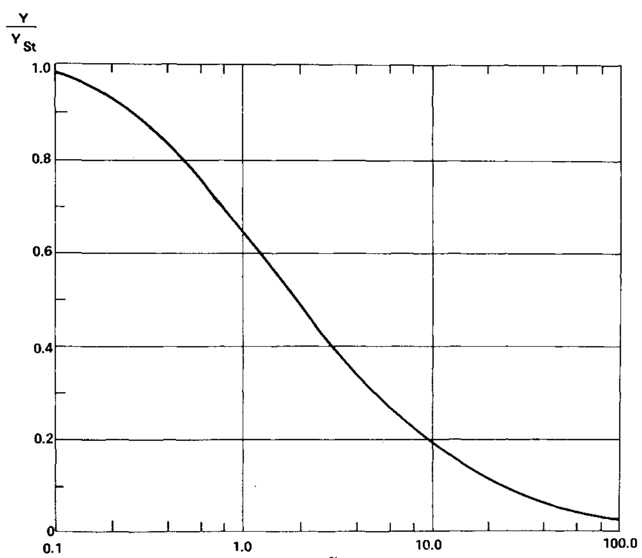

Fig. 1 Ratio of particle penetration for "standard" drag to that for Stokes drag.

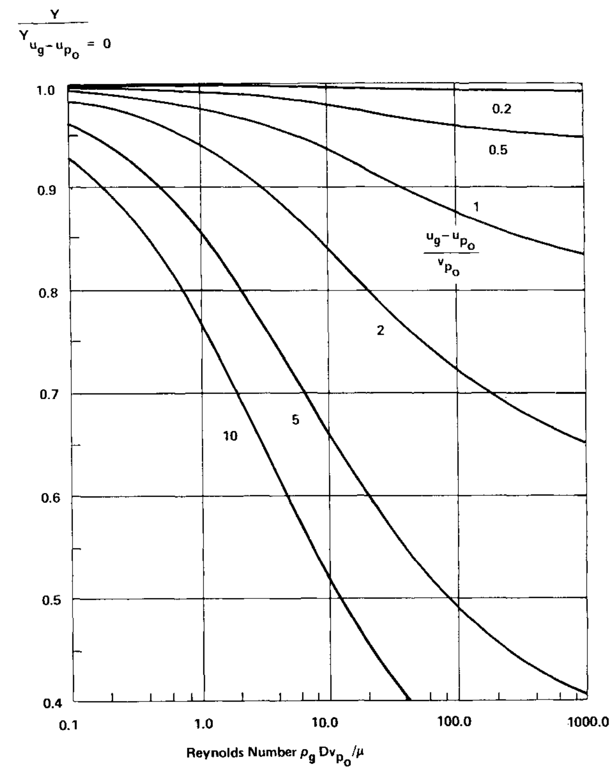

Fig. 2 Ratio of particle penetration into a cross flow to that without cross flow.

The first group of variables in the last equation represents the particle Reynolds number based on the initial lateral velocity component. Equation (7) is readily integrated by taking $(v_p/v_{p_o})^{1/3}$ as a new independent variable. This integration, with consideration that $v_p = v_{p_o}$ for $y = 0$, leads to

$$y = \frac{\rho_p D^2 v_{p_o}}{18\mu}\frac{3}{\alpha}\left[1-\left(\frac{v_p}{v_{p_o}}\right)^{1/3}+\frac{\tan^{-1}[\alpha^{1/2}(v_p/v_{p_o})^{1/3}]}{\alpha^{1/2}}-\frac{\tan^{-1}\alpha^{1/2}}{\alpha^{1/2}}\right] \tag{9}$$

Since $v_p = 0$ at the point of maximum penetration, $y_{max} = Y$, Eq. (9) yields

$$Y = (\rho_p D^2 v_{p_o}/18\mu)F(\alpha) \tag{10}$$

where

$$F(\alpha) = (3/\alpha)[1-(\tan^{-1}\alpha^{1/2}/\alpha^{1/2})] \tag{11}$$

It follows from the derivation of Eq. (7) that the assumption of Stokes drag instead of the "standard" drag given by Eq. (1) is equivalent to setting $\alpha = 0$. In this case, Eqs. (10) and (11) yield the well-known penetration of a particle obeying Stokes drag

$$Y_{st} = \rho_p D^2 v_{p_o}/(18\mu) \tag{12}$$

which is independent of the cross flow as mentioned earlier.

The ratio, $Y/Y_{st} = F(\alpha)$, is plotted in Fig. 1 which indicates that the penetration computed on the basis of Stokes drag can be much too large. The effect of cross flow on the penetration is given by the ratio

$$Y/Y_{u_g-u_{p_o}=0} = F(\alpha)/F(\alpha_{u_g-u_{p_o}=0})$$

This ratio can be computed from Eqs. (8) and (11) and is plotted in Fig. 2 as a function of the Reynolds number, $\rho_g D v_{p_o}/\mu$, for several values of the velocity ratio, $(u_g-u_{p_o})/v_{p_o}$.

Discussion

The method used to derive Eq. (9) also could be used to obtain the relationship between the longitudinal particle velocity and coordinate. This development is straightforward and not carried out here.

The effect of cross flow can be a significant reduction of the penetration of a particle, especially if the initial longitudinal velocity difference, $u_g - u_{p_o}$, exceeds the lateral injection velocity.

As an illustrative example, consider a 100-μm particle having a density of 2.5 g/cm^3 injected laterally ($u_{p_o} = 0$) with a velocity of 10 m/sec into an air flow of 20 m/sec and atmospheric density ($\rho_g = 1.2 \times 10^{-3}$ g/cm^3, $\mu = 1.8 \times 10^{-4} P$). For these data, the Reynolds number based on v_{p_o} is 67 and $\alpha = 4.7$. Equation (12) yields $Y_{st} = 77$ cm and Fig. 1 shows that $Y/Y_{st} = 0.3$. The actual penetration therefore is only 23 cm. Figure 2 indicates that $Y/Y_{u_g - u_{p_o} = 0} = 0.735$; penetration without cross flow therefore would be 31 cm.

Calculation of particle penetration often is based on Stokes drag because of the resultant simplification of the equations. As the foregoing example shows, penetrations computed in this manner may be much too large because Stokes drag is too low and the effect of cross flow is neglected. The present analysis provides a convenient method for the calculation of particle penetration without having to resort to the unrealistic assumption of Stokes drag.

Particles injected by a carrier gas are directly affected by the cross flow only after their inertia carries them into the main flow. The early part of the "free-particle" trajectory therefore is influenced by the carrier gas. This effect can be expected to be of little importance for large particles but may be significant for small particles. Proper assessment of this problem requires knowledge of the behavior of gas-particle jets in a cross flow. Such studies are currently in progress.

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Transient Molecular Concentration Measurements in Turbulent Flows Using Rayleigh Light Scattering

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Introduction

THE current trend in turbulence research is away from a preoccupation with simple time averages and towards the investigation of detailed time histories of the flow variables. This follows from the recognition that many turbulent flows are dominated by relatively well-ordered structures. Vitally important to this development is the ability to make reliable time and space-resolved measurements of composition and temperature in the turbulent field.

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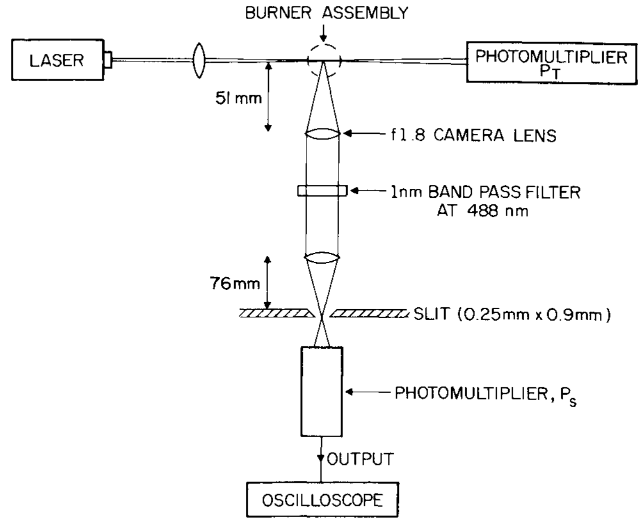

Fig. 1 Details of the optical system (plan view).

Molecular Rayleigh and Raman scattering are well suited to obtain local instantaneous measurement of individual gas densities. They are particularly advantageous in the study of hostile or inaccessible environments, and also when the presence of a physical probe would cause an unacceptable perturbation to the system under study. In this Note we discuss briefly the range of systems where Rayleigh rather than Raman scattering is of the greatest use and describe its successful application to the study of lifted jet diffusion flames. This particular regime has been chosen since we have recently shown,[1] using a gated mass-spectrometric probe technique, that the flow is dominated by the phenomenon of "unmixedness."[2] This technique does, however, still produce a certain element of time-averaging as well as physically disturbing the flow. It is also limited to the study of exactly periodic flows. Up to the present, Rayleigh (elastic) scattering‡ from gas flows has received far less attention than Raman (inelastic) scattering. This is because the frequency shift of a Raman transition (usually the Q branch of a vibrational band) together with the intensity and a polarization-dependent ratio serves in most cases to identify uniquely the chemical species concerned. Thus Vibrational Raman (VR) scattering measurements can be used to determine the concentrations of individual species in a multicomponent system. In contrast, Rayleigh scattering measurements can provide, at any instant, only a single intensity and a polarization ratio, no matter how many species are present, and thus their usefulness is restricted to simpler systems. However, the scattering cross sections for VR transitions are typically three orders of magnitude lower than the corresponding Rayleigh cross sections. Thus, with currently available light sources, measurements of transient concentrations by VR scattering (at atmospheric pressure) cannot be obtained with a time resolution better than about 1 sec. This is inadequate for studying the time-dependence of concentrations in turbulent flows, whereas a time resolution of better than 1 msec, which can be obtained from Rayleigh scattering measurements, is quite sufficient to resolve much of the turbulence structure.

General Principles

Molecular Rayleigh scattering is characterized by two variables, commonly chosen to be a total scattering cross section and the eccentricity of the polarizability ellipsoid. In this work, however, we choose the variables to be the differential scattering

‡ Throughout this paper Rayleigh scattering intensities and cross sections are assumed to include contributions from the entire rotational Raman spectrum and not just the transitions with $\Delta J = 0$.